# Remainder and Factor Theorem

**IP/O Level Math**

## Topic Summary

| Definition of Polynomial |
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| A **polynomial** is a mathematical expression of one or sum of algebraic terms each of which is either a constant term or a variable term with whole number exponent. |
| **Degree** |
| The **degree** of a polynomial in one variable is the highest exponent of the variable term in the polynomial. |

## Solving problems related to equality of polynomials

### Comparing coefficients

Example:

Let $\mathrm{P}(x)=6x^3-14x^2+19x-5$ and let $\mathrm{Q}(x)$ be some polynomial. Find the constants $A$, $B$ and $C$, where appropriate, in the following equation.

(a) $\mathrm{P}(x)=(3x-1)\left(Ax^2+Bx+C\right)$

Workings:

$(3x-1)(Ax^2+Bx+C) = 3Ax^3+3Bx^2+3Cx-Ax^2-Bx-C$ [expansion]

$= 3Ax^3+(3B-A)x^2+(3C-B)x-C$ [group like terms]

By comparison, [compare the expanded term with the given P(x) from the question]

3A = 6

A = 2

3B-A = -14

B = -4

C = 5

### Numerical substitution

Example:

Let $\mathrm{P}(x)=6x^3-14x^2+19x-5$ and let $\mathrm{Q}(x)$ be some polynomial. Find the constants $A$, $B$ and $C$, where appropriate, in the following equation.

(b) $\mathrm{P}(x)=(x-1)(x-2)\mathrm{Q}(x)+Ax+B$

Workings:

Let $Q(x) = Cx - D$ [substitution]

$(x-1)(x-2)(Cx-D)+Ax+B = Cx^3-3Cx^2+2Cx+Dx^2-3DX+2D+Ax+B$

$= Cx^3+(D-3C)x^2+(2C-3D+A)x+2D+B$ [expand and group like terms]

By comparison, [compare the expanded term with the given P(x) from the question]

C = 6

D - 3C = -14

D = 4

2D + B = -5

B = -13

2C – 3D + A = 19

A = 19

| **Long division for polynomials** | | |
|---|---|---|
| • We carry out long division only to **improper fraction** (of polynomials) – A fraction whose numerator is of degree greater or equal to the denominator.<br>• The remainder always has a degree lower than the divisor. | | |
| Example: Divide the expression $3x^2+5-2x$ by $x+3$. | | |
| STEP 1 | Write the dividend $3x^2-2x+5$ and the divisor x + 3 in the division format, in descending powers of $x$. | $x+3\overline{)3x^2-2x+5}$ |
| STEP 2 | Identify the term 3x such that multiply it to x + 3 gives the highest power term $3x^2$ in the dividend. Write $3x$ in the quotient position. | $\begin{array}{r} 3x \\ x+3\overline{)3x^2-2x+5} \end{array}$ |
| STEP 3 | Multiply $3x$ by $x+3$ and write the answer below the dividend, align the same power terms in the same column. | $\begin{array}{r} 3x \\ x+3\overline{)3x^2-2x+5} \\ 3x^2+9x \end{array}$ |
| STEP 4 | Subtract the polynomial $3x^2-2x+5$ with $3x^2+9$ and write the answer below in the next line, aligning the same power terms in the same column. | $\begin{array}{r} 3x \\ x+3\overline{)3x^2-2x+5} \\ (-)\ 3x^2+9x \\ \hline -11x+5 \end{array}$ |
| Repeat STEP 2 to STEP 4 till the answer obtained has a lower degree than the divisor. | | |
| STEP 5-2 | Identify the next monomial as $-11$ because $-11(x+3)=-11x\cdots$ which matches the highest power term in $-11x+5$. Write $-11$ after $3x$ in the quotient position as shown. | $\begin{array}{r} 3x-11 \\ x+3\overline{)3x^2-2x+5} \\ 3x^2+9x \\ \hline -11x+5 \end{array}$ |
| STEP 5-3 | Multiply the monomial $-11$ by the divisor $x+3$ and write the answer below the dividend, align the same power terms in the same column. | $\begin{array}{r} 3x-11 \\ x+3\overline{)3x^2-2x+5} \\ 3x^2+9x \\ \hline -11x+5 \\ -11x-33 \end{array}$ |
| STEP 5-4 | Subtract $-11x+5$ with $-11x-33$ and obtain $38$ whose power is lower than that of the divisor. The division process stops.<br><br>The term $38$ is called the remainder and $3x-11$ the quotient.<br><br>Finally, we have $\dfrac{3x^2-2x+5}{x+3}=3x-11+\dfrac{38}{x+3}$. | $\begin{array}{r} 3x-11 \\ x+3\overline{)3x^2-2x+5} \\ 3x^2+9x \\ \hline -11x+5 \\ (-)\ -11x-33 \\ \hline 38 \end{array}$ |

| Remainder Theorem |
|---|
| The remainder theorem allows us to find the remainder without doing long division.<br>When a polynomial $\mathrm{P}(x)$ is divided by a linear divisor $ax+b$, the remainder is $\mathrm{P}\left(-\frac{b}{a}\right)$. |
| **Factor Theorem** |
| A polynomial $\mathrm{P}(x)$ has a factor $ax+b$ if and only if $\mathrm{P}\left(-\frac{b}{a}\right)=0$.<br>We can also say that $\mathrm{P}(x)$ is **divisible** by $ax+b$.<br>In terms of equation, we say that $x=-\frac{b}{a}$ is a **root** of $\mathrm{P}(x)=0$. |

**Some special cases when $\mathrm{P}(x)$ is divided by $\mathrm{D}(x)$**

| Divisor D($x$) | Remainder R($x$) | Method to find R($x$) |
|---|---|---|
| $x-a$ | $\mathrm{P}(a)$ | Substitute $x = a$ into P($x$).<br>Remainder = P($a$).<br><br>OR<br><br>*Use long division to divide $\mathrm{P}(x)$ by $x-a$. |
| $(x-a)(x-b)$ in factorised form | $Ax+B$ | Let R($x$) = $Ax + B$<br>Substitute $x=a$: $\mathrm{P}(a)=aA+B$ and substitute $x=b$: $\mathrm{P}(b)=bA+B$. Solve for $A$ and $B$ to find the remainder.<br><br>OR<br><br>*Expand $(x-a)(x-b)$ and use long division to divide $\mathrm{P}(x)$ by $(x-a)(x-b)$. |
| $x^2+cx+d$ cannot be factorised further | $Ax+B$ | *Use long division to divide $\mathrm{P}(x)$ by $x^2+cx+d$ to find the remainder. |

- Note that use of long division is complicated or not advisable if the polynomial (dividend) contains one or more unknown coefficients.

- To overcome polynomial with unknown coefficients, write $P(x) = D(x)Q(x) + R(x)$ and use the **method of numerical substitution**.

# Remainder and Factor Theorem Practice Questions

1. When $2x^4 - 6x^2 + 3x + 1$ is divided by $x^2 - 3x - 4$, the remainder is $Ax + B$.
   (a) Find the values of the constants $A$ and $B$.
   (b) Find the remainder when the polynomial is divided by $x^2 + 2x + 1$.

2. Solve the equation $2x^3 - 5x^2 + 4 = 0$, leaving your answers in exact form.

3. Solve the equation $2x^3 - 9x^2 + 11x - 2 = 0$ giving your answers correct to 2 decimal places where necessary. [4]
   Hence solve the following equations
   (a) $2(y + 2)^3 - 9(y + 2)^2 + 11(y + 2) - 2 = 0$,
   (b) $2z^3 + 9z^2 + 11z + 2 = 0$.

4. One of the factors of $x^2 - 5x + 6$ is a factor of $f(x) = x^3 + hx^2 - 4x + 3$ where $h$ is an integer. Find the common factor and the value of $h$, hence find the remainder when $f(x)$ is divided by $3x + 2$.

**Answers**

1. (a) $A = 87$ , $B = 81$ (b) $7x + 1$

2. $x = 2, \frac{1}{4} \pm \frac{1}{4}\sqrt{17}$

3. $x$ = 2, 2,28, 0.22 (a) $y$ = 0, –1.78, 0.28 (b) $z$ = –2, –2,28, –0.22

4. common factor = $x - 3$ , $h = -2$ ; $4\frac{13}{27}$