Differentiation

JC A Level Math

Topic Summary

Techniques of Differentiation		
y	$\frac{dy}{dx}$	**Remarks**
a	0	a, n and c are constants.
x^n	nx^{n-1}	
$f(x) \pm g(x)$	$\frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$	
uv, where u, v are functions of x	$u\frac{dv}{dx}+v\frac{du}{dx}$	Product Rule
$\frac{u}{v}$ where u, v are functions of x	$\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$	Quotient Rule
$f(g(x))$	If $y=f(g(x))$ and $u=g(x)$, then $y=f(u)$ and $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$	Chain Rule

Differentiation of Logarithm and Exponential Functions	
y	$\frac{dy}{dx}$
$\log_a x$	$\frac{1}{x \ln a}$
$\log_a \mathrm{f}(x)$	$\frac{1}{\mathrm{f}(x) \ln a} \mathrm{f}'(x)$
$\ln x$	$\frac{1}{x}$
$\ln\left[\mathrm{f}(x)\right]$	$\frac{1}{\mathrm{f}(x)} \mathrm{f}'(x)$
a^x	$a^x \ln a$
$a^{\mathrm{f}(x)}$	$\left[a^{\mathrm{f}(x)} \ln a\right] \cdot \mathrm{f}'(x)$
e^x	e^x
$\mathrm{e}^{\mathrm{f}(x)}$	$\mathrm{e}^{\mathrm{f}(x)} \mathrm{f}'(x)$

Differentiation of Trigonometric Functions	
y	$\frac{dy}{dx}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Differentiation of Inverse Trigonometric Functions	
y	**$\dfrac{dy}{dx}$**
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}, \quad \lvert x \rvert < 1$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}, \quad \lvert x \rvert < 1$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}, \quad x \in \mathbb{R}$

Implicit Differentiation

$$\frac{d}{dx} g(y) = \left[\frac{d}{dy} g(y)\right] \frac{dy}{dx}$$

Higher Order Differentiation

$$\frac{d^n y}{dx^n} = f^{(n)}(x) = \frac{d}{dx}\left(\frac{d^{n-1} y}{dx^{n-1}}\right)$$

Parametric Differentiation

If x and y are functions of the parameter t, then

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}.$$

Differentiation Practice Questions

1. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = \lg \sqrt[4]{4\cos^2\left(2x^2 - 1\right) - 3\sec\left(4x^3\right)}$.

2. The variables x and y are related by $\mathrm{e}^{xy^2} = y\left(x^2 + 2\mathrm{e}^x\right)$. Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 0$ and $y = \frac{1}{2}$.

3. A curve has the parametric equations $x = \frac{t}{a+t}$, $y = \frac{t^2}{a+t}$, where a is a non-zero constant. Find the gradient of the curve at the point with parameter ap.

4. [2014/VJC/Prelim/1/10]
The curve C with equation $y = \cos^{-1}\left(\frac{1}{x}\right)$ for $0 \le y \le \frac{\pi}{3}$ is rotated about the y-axis to form the curved surface of an open container with a horizontal circular base of radius 1 unit. Given that $0 \le h \le \frac{\pi}{3}$, show that V, the volume of water inside the container when it is used to hold water up to a depth of h, is given by $V = \pi \tan h$.

   The container was initially full of water but a hole at the base causes water to leak at a rate inversely proportional to h, the depth of water at time t minutes. By first forming an equation relating $\frac{\mathrm{d}V}{\mathrm{d}t}$, $\frac{\mathrm{d}h}{\mathrm{d}t}$ and h, show that $h\sec^2 h\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{k}{\pi}$, where k is a positive constant.
Given that it takes half an hour for the water to completely drain out, find the value of k.

5. A hot kettle at the initial temperature 200 °C is put in a room at the temperature of 20 °C. The temperature of the kettle, θ °C, decreases according to the Newton's law of cooling which states that the rate of cooling is proportional to the difference of temperatures between the heated body and the environment. The temperature of the room increases at the rate of 2 °C per minute.

   (i) Show that θ and t are connected by the differential equation $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k\left(\theta - 20 - 2t\right)$, where t is the time in minutes after the kettle is put in the room.

   (ii) Use the substitution $u = \theta - 20 - 2t$ to solve the equation in part (i).

   (iii) Given that $k = \frac{1}{6}$ per minute, find the time when the kettle's temperature and the room temperature become equal.

Answers

1. $$\frac{8x\cos\left(2x^2-1\right)\sin\left(2x^2-1\right)+9x^2\sec\left(4x^3\right)\tan\left(4x^3\right)}{\ln 10\left[3\sec\left(4x^3\right)-4\cos^2\left(2x^2-1\right)\right]}$$

2. $-\frac{3}{8}$

3. $ap\left(p+2\right)$

4. 0.117

5. (ii) $\theta=20-\frac{2}{k}+2t+\left(180+\frac{2}{k}\right)e^{-kt}$ (iii) 16.6 minutes