Probability

JC A Level Math

Topic Summary

- If the sample space S consists of a finite number of equally likely outcomes, then $\mathrm{P}(A)=\dfrac{\mathrm{n}(A)}{\mathrm{n}(S)}$, where n($A$) denotes the number of outcomes in the event A.
- For complementary event, $\mathrm{P}(A') = 1 - \mathrm{P}(A)$.
- Common useful results from Venn diagram:

  $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

  $\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B')$

  $\mathrm{P}(A' \cap B') = 1 - \mathrm{P}(A \cup B)$
- Conditional Probability: $\mathrm{P}(A|B)=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$ where $\mathrm{P}(B) \neq 0$
- Two events A and B are mutually exclusive if $A \cap B = \varnothing$, i.e., $\mathrm{P}(A \cap B) = 0$.
- To show that two events A and B are independent, show either

$\mathrm{P}(A|B)=\mathrm{P}(A)$ or

$\mathrm{P}(B|A)=\mathrm{P}(B)$ or

$\mathrm{P}(A)\,\mathrm{P}(B) = \mathrm{P}(A \cap B)$.

Probability Practice Questions

1. **N2004/II/23 & N2001/II/11(a)**

(a) A and B are events such that $\mathrm{P}(A \cup B) = 0.9$, $\mathrm{P}(A \cap B) = 0.2$ and $\mathrm{P}(A|B) = 0.8$. Find $\mathrm{P}(A)$ and $\mathrm{P}(B')$.

(b) Events A and B are such that $\mathrm{P}(A) = \frac{1}{3}$, $\mathrm{P}(B|A) = \frac{1}{4}$ and $\mathrm{P}(A' \cap B') = \frac{1}{6}$. Find

(i) $\mathrm{P}(A \cup B)$, (ii) $\mathrm{P}(B)$.

For both (a) and (b), determine if the two events A and B are independent.

2. **N2010/II/7**

For events A and B it is given that $\mathrm{P}(A) = 0.7$, $\mathrm{P}(B) = 0.6$ and $\mathrm{P}(A \mid B') = 0.8$. Find

(i) $\mathrm{P}(A \cap B')$,

(ii) $\mathrm{P}(A \cup B)$,

(iii) $\mathrm{P}(B' \mid A)$.

For a third event C, it is given that $\mathrm{P}(C) = 0.5$ and that A and C are independent.

(iv) Find $\mathrm{P}(A' \cap C)$.

(v) Hence state an inequality satisfied by $\mathrm{P}(A' \cap B \cap C)$.

3. **N93/II/6(a)**

Two fair dice are thrown, and events A, B and C are defined as follows:

A: the sum of the two scores is odd,

B: at least one of the two scores is greater than 4,

C: the two scores are equal.

Find, showing your reasons clearly in each case, which two of these three events are (i) mutually exclusive, (ii) independent.

Find also $P(C|B)$.

4. **N2002/II/30 (a)**

A room contains n randomly chosen people.

(a) Assume that a randomly chosen person is equally likely to have been born on any day of the week. The probability that the people in the room were all born on different days of the week is denoted by P.

(i) Find P in the case $n = 3$.

(ii) Show that $P = \frac{120}{343}$ in the case $n = 4$.

(b) Assume now that a randomly chosen person is equally likely to have been born on any month of the year. Find the smallest value of n such that the probability that the people in the room were all born on different months of the year is less than ½ .

(c) Assume now that a randomly chosen person is equally likely to have been born on any day of the 365 days in the year. It is given that, for the case $n = 21$, the probability that the people in the room were all born on different days of the year is 0.55631, correct to 5 decimals. Find the smallest value of n such that the probability that at least two people were born on the same day of the year exceeds ½.

Answers

1. (a) 0.85, 0.75, not independent (b)(i) $\frac{5}{6}$ (ii) $\frac{7}{12}$, not independent

2. (i) 0.32 (ii) 0.92 (iii) 0.457
   (iv) 0.15 (v) $P(A' \cap B \cap C) \leq 0.15$

3. (i) A and C (ii) A and B, $\frac{1}{10}$

4. (a) (i) $\frac{30}{49}$ (b) 5 (c) 23

Permutations and Combinations

JC A Level Math

Topic Summary

Basic Counting Principles	
Addition Principle Consider mutually exclusive cases	**Multiplication Principle** Consider consecutive operations one after another

Arrangement: Order is important	
Types of arrangement	Formulae
Permutation of n distinct objects	$n!$
Permutation of r objects taken from n distinct objects, without replacement	${}^nP_r = \frac{n!}{(n-r)!}$
Arrangement of r objects taken from n distinct objects, with replacement	n^r
Arrangement of n objects, not all distinct	$\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_k!}$
Permutation of n distinct objects in a circle	$\frac{n!}{n} = (n-1)!$

Combination (or Selection): Order is not important	
Type of combination	Formula
n distinct objects, taken r at a time	${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

Relationship between ${}^nP_r$ and ${}^nC_r$: ${}^nP_r = {}^nC_r \times r!$

Useful techniques/methods for solving counting problems:

- Take note of restrictions or constraints
- Consider objects in sequence
- Method of 'Complementation'
- Method of 'Grouping'
- Method of 'Slotting'

Permutations and Combinations Practice Questions

1. **[N2009/P2/8]**

Find the number of ways in which the letters of the word ELEVATED can be arranged if

(i) there are no restrictions,

(ii) T and D must not be next to one another,

(iii) consonants (L, V, T, D) and vowels (E, A) must alternate,

(iv) between any two Es there must be at least 2 other letters.

2. **[PJC07/BT/P2/7]**

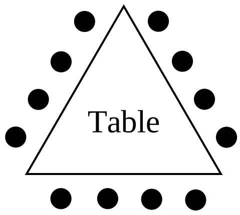
(a) Peter attempts to break a four-digit code, which is made up of numbers 1, 2,…, 9. Given that the digits may be repeated, in how many ways can the code be formed if

(i) the last digit of the code is a ‘3’,

(ii) the code is an odd-number greater than 6000,

(b) The number 1086624 can be expressed in prime factors as $2^5 \times 3^2 \times 7^3 \times 11^1$. Excluding 1 and 1086624, how many positive integers are factors of 1086624?

3. A group of 12 people which consists of 6 married couples sits at a triangular table, which is equilateral, with 4 persons at each side of the table as shown:



(i) Find the number of different possible arrangements.

(ii) Find the number of different possible arrangements if each man sits next to his wife, with both husband and wife on the same side of the table.

4. **[TJC08/Prelim/P2/5]**

Farmers Alex, Ben and Charlie are discussing their plans for a piece of land which is divided into 9 plots as numbered in the diagram below.

1	2	3
4	5	6
7	8	9

(i) Farmer Alex wishes to plant wheat in 2 even-numbered plots and corn in 3 odd-numbered plots. How many ways are there for him to do so?

(ii) A cluster is a set of 4 adjacent plots which form a square. For example, plots 1, 2, 4 and 5 form a cluster. Farmer Ben wishes to plant 4 different types of fruits in a cluster, each in a different plot. How many ways are there for him to do so?

(iii) Farmer Charlie wishes to select 4 non-adjacent plots in which to plant 4 different types of vegetables, each in a different plot. For example, plots 1 and 2 are adjacent, but plots 1 and 5 are not. How many ways are there for him to do so?

Answers

1. (i) 6720 (ii) 5040 (iii) 192 (iv) 480

2. (a)(i) 729 (ii)1620 (b) 142

3. (i) 159667200 (ii) 15360

4. (i) 60, (ii) 96, (iii) 144